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An introduction to the theory of Hom–Hilbert algebras

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ABSTRACT. The concept of (commutative, bounded) Hom-Hilbert algebras is introduced, and examples are provided to illustrate it. Fundamental properties of the (commutative and bounded) Hom-Hilbert algebra are investigated, and a method for deriving the generalized Hilbert algebra from the Hom-Hilbert algebra is presented, and vice versa. A process is given in which every commutative Hom-Hilbert algebras form a poset structure. The join-semilattice is induced based on the commutative Hom-Hilbert algebra, and the meet-semilattice is induced based on the bounded Hom-Hilbert algebra. The commutative residuated lattice based on the bounded commutative Hom-Hilbert algebra is constructed.

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1. INTRODUCTION

Hom-type algebras are generalizations of classical algebraic structures (like associative algebras, Lie algebras, groups, etc.) in which the defining identities are “twisted” by a linear self-map, typically denoted α. As an application of Hom-type algebras, it appears primarily in mathematical physics as an algebraic framework for quantum deformations and twisted symmetries in mathematical physics, and as generalized structures in Poisson/symplectic and algebraic geometry. In algebraic structures, the Hom-type theory is applied to groups, Lie-algebras, etc. (See [1, 2, 3, 4, 5]).

This paper focuses on the study of Hom-type algebras on generalized Hilbert algebras. We are going to deal with the following items.

1. Making the (commutative, bounded) Hom-Hilbert algebra: We define the concept of (commutative, bounded) Hom-Hilbert algebras.
2. Finding examples for the (commutative, bounded) Hom-Hilbert algebra: We give examples to illustrates the (commutative, bounded) Hom-Hilbert algebra.
3. Examining the fundamental properties of the (commutative and bounded) Hom-Hilbert algebra: We examine several properties on the (commutative, bounded) Hom-Hilbert algebra.
4. Suggesting a way to induce the Hom-Hilbert algebra from the generalized Hilbert algebra, and vice versa: We suggest a way to derive a generalized Hilbert algebra from a Hom-Hilbert algebra, and vice versa.
5. Forming the poset, semilattice structures, and residuated lattice from a Hom-Hilbert algebra: We show that every commutative Hom-Hilbert algebra forms the poset structure. We induce the join-semilattice based on the commutative Hom-Hilbert algebra, and we induce the meet-semilattice based on the bounded Hom-Hilbert algebra. We construct the commutative residuated lattice based on the bounded commutative Hom-Hilbert algebra.

2. PRELIMINARIES

A *Hilbert algebra* (See [6]) is defined as a triple $(H, *, 1)$, where $*$ is a binary operation and 1 is a constant, satisfying the following three axioms for all $x, y, z \in H$:

(HA1) $x * (y * x) = 1$,
(HA2) $(x * (y * z)) * ((x * y) * (x * z)) = 1$,
(HA3) $x * y = 1, y * x = 1 \implies x = y$.

A *generalized Hilbert algebra* (briefly, *g-Hilbert algebra*) (See [7]) is defined as a triple $(H, *, 1)$ where $*$ is a binary operation and 1 is a constant, satisfying the following three axioms for all $x, y, z \in H$:

(gH1) $1 * x = x$,
(gH2) $x * x = 1$,
(gH3) $x * (y * z) = y * (x * z)$,
(gH4) $x * (y * z) = (x * y) * (x * z)$.

Every Hilbert algebra is a g-Hilbert algebra, but not vice vresa (See [7]).

A g-Hilbert algebra $(H, *, 1)$ is said to be *commutative* (See [7]), if it satisfies:

$$(\forall x, y \in H)((x * y) * y = (y * x) * x). \tag{2.1}$$

3. HOM-HILBERT ALGEBRAS

As a Hom-type generalization of a classical g-Hilbert algebra, we introduce the concept of Hom-Hilbert algebras.

Definition 3.1. A *Hom-Hilbert algebra* is a quadruple $\mathcal{H} := (H, \to, 1, \alpha)$, where H is a nonempty set, $\to$ is a binary operation on H, $1 \in H$ is the unit and α is a self-map (called the twisting map) of H, satisfying the following axioms for all $x, y, z \in H$:

(HH1) $\alpha(1) \to x = x$,
(HH2) $\alpha(x) \to x = 1$,
(HH3) $\alpha(x) \to (\alpha(y) \to z) = \alpha(y) \to (\alpha(x) \to z)$,

(HH4) $\alpha(x) \to (\alpha(y) \to z) = \alpha(\alpha(x) \to y) \to (\alpha(x) \to z)$.

Example 3.2. (1) Let $H = \{1, a, b\}$ be a set with the Cayley table for $\to$:

$\to$	1	a	b
1	1	a	b
a	1	1	1
b	1	a	1

Define the twisting map $\alpha : H \to H$ by $\alpha(1) = 1$, $\alpha(a) = a$ and $\alpha(b) = a$. It is routine to check that $\mathcal{H} := (H, \to, 1, \alpha)$ is a Hom-Hilbert algebra.

(2) Let H be the infinite real interval $[0, 1]$ and consider the Gödel implication $\to$, that is,

$$(\forall x, y \in H)\left(x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}\right).$$

Define the twisting map

$$\alpha : H \to H,\ x \mapsto \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x < 1. \end{cases}$$

Then $\mathcal{H} := (H, \to, 1, \alpha)$ is a Hom-Hilbert algebra.

Define a binary relation $\leq_\alpha$ by

$$(\forall x, y \in H)(x \leq_\alpha y \iff \alpha(x) \to y = 1).$$

Proposition 3.3. *Every Hom-Hilbert algebra $\mathcal{H} := (H, \to, 1, \alpha)$ satisfies the following conditions: for all $x, y, z \in H$,*

$$x \leq_\alpha x,\ x \leq_\alpha 1, \tag{3.1}$$

$$x \leq_\alpha y,\ y \leq_\alpha z \implies x \leq_\alpha z, \tag{3.2}$$

$$y \leq_\alpha z \implies \alpha(x) \to y \leq_\alpha \alpha(x) \to z, \tag{3.3}$$

$$x \leq_\alpha \alpha(y) \to x, \tag{3.4}$$

$$x \leq_\alpha y \implies \alpha(y) \to z \leq_\alpha \alpha(x) \to z, \tag{3.5}$$

$$\alpha(x) \to y \leq_\alpha \alpha(\alpha(y) \to z) \to (\alpha(x) \to z). \tag{3.6}$$

Proof. The axiom (HH2) immediately means $x \leq_\alpha x$. If we put $y = x$ and $z = x$ in (HH4), then

$$\begin{aligned} \alpha(x) \to 1 &\overset{\text{(HH2)}}{=} \alpha(x) \to (\alpha(x) \to x) = \alpha(\alpha(x) \to x) \to (\alpha(x) \to x) \\ &\overset{\text{(HH2)}}{=} \alpha(1) \to 1 \overset{\text{(HH1)}}{=} 1. \end{aligned}$$

Thus $x \leq_\alpha 1$. So (3.1) is valid.

Suppose $x \leq_\alpha y$ and $y \leq_\alpha z$. Then $\alpha(x) \to y = 1$ and $\alpha(y) \to z = 1$. Thus

$$\begin{aligned}\alpha(x) \to z &\overset{(\text{HH1})}{=} \alpha(1) \to (\alpha(x) \to z)\\ &= \alpha(\alpha(x) \to y) \to (\alpha(x) \to z)\\ &\overset{(\text{HH4})}{=} \alpha(x) \to (\alpha(y) \to z)\\ &= \alpha(x) \to 1 \overset{(3.1)}{=} 1.\end{aligned}$$

So $x \leq_\alpha z$, which shows (3.2).

Suppose $y \leq_\alpha z$. Then $\alpha(y) \to z = 1$. Thus

$$\alpha(\alpha(x) \to y) \to (\alpha(x) \to z) \overset{(\text{HH4})}{=} \alpha(x) \to (\alpha(y) \to z) = \alpha(x) \to 1 \overset{(3.1)}{=} 1.$$

So $\alpha(x) \to y \leq_\alpha \alpha(x) \to z$, which proves (3.3). For every $x, y \in H$, we have

$$\alpha(x) \to (\alpha(y) \to x) \overset{(\text{HH3})}{=} \alpha(y) \to (\alpha(x) \to x) \overset{(\text{HH2})}{=} \alpha(y) \to 1 \overset{(3.1)}{=} 1.$$

Hence $x \leq_\alpha \alpha(y) \to x$ which proves (3.4).

Suppose $x \leq_\alpha y$. then $\alpha(x) \to y = 1$. Thus

$$\begin{aligned}\alpha(x) \to (\alpha(y) \to z) &\overset{(\text{HH4})}{=} \alpha(\alpha(x) \to y) \to (\alpha(x) \to z)\\ &= \alpha(1) \to (\alpha(x) \to z)\\ &\overset{(\text{HH1})}{=} \alpha(x) \to z.\end{aligned}$$

It follows that

$$\begin{aligned}\alpha(\alpha(y) \to z) \to (\alpha(x) \to z) &= \alpha(\alpha(y) \to z) \to (\alpha(x) \to (\alpha(y) \to z))\\ &\overset{(\text{HH3})}{=} \alpha(x) \to (\alpha(\alpha(y) \to z) \to (\alpha(y) \to z))\\ &\overset{(\text{HH2})}{=} \alpha(x) \to 1 \overset{(3.1)}{=} 1.\end{aligned}$$

So $\alpha(y) \to z \leq_\alpha \alpha(x) \to z$, i.e., (3.5) is valid. For every $x, y, z \in H$, we have

$$\begin{aligned}&\alpha(\alpha(x) \to y) \to (\alpha(\alpha(y) \to z) \to (\alpha(x) \to z))\\ &\overset{(\text{HH3})}{=} \alpha(\alpha(y) \to z) \to (\alpha(\alpha(x) \to y) \to (\alpha(x) \to z))\\ &\overset{(\text{HH4})}{=} \alpha(\alpha(y) \to z) \to (\alpha(x) \to (\alpha(y) \to z))\\ &\overset{(\text{HH3})}{=} \alpha(x) \to (\alpha(\alpha(y) \to z) \to (\alpha(y) \to z))\\ &\overset{(\text{HH2})}{=} \alpha(x) \to 1 \overset{(3.1)}{=} 1,\end{aligned}$$

Hence $\alpha(x) \to y \leq_\alpha \alpha(\alpha(y) \to z) \to (\alpha(x) \to z)$, i.e., (3.6) is valid. □

We can see that the binary relation $\leq_\alpha$ is a preorder on H with 1 as greatest element (See (3.1) and (3.2)).

In Example 3.2, we have

$$\leq_\alpha = \{(1,1), (a,1), (a,a), (a,b), (b,1), (b,a), (b,b)\}.$$

Then $a \leq_\alpha b$ and $b \leq_\alpha a$, but $a \neq b$. Thus the binary relation $\leq_\alpha$ does not satisfy the antisymmetry.

The theorem below is a way to derive a g-Hilbert algebra from a Hom-Hilbert algebra.

Theorem 3.4. *Let* $\mathcal{H} := (H, \to, 1, \alpha)$ *be a Hom-Hilbert algebra. If we define* $x * y = \alpha(x) \to y$ *for* $x, y \in H$*, then* $(H, *, 1)$ *is a* g*-Hilbert algebra.*

Proof. Let $x, y, z \in H$. Then $1 * x = \alpha(1) \to x \overset{\text{(HH1)}}{=} x$, $x * x = \alpha(x) \to x \overset{\text{(HH2)}}{=} 1$,

$$\begin{aligned} x * (y * z) &= \alpha(x) \to (y * z) = \alpha(x) \to (\alpha(y) \to z) \\ &\overset{\text{(HH3)}}{=} \alpha(y) \to (\alpha(x) \to z) \\ &= \alpha(y) \to (x * z) = y * (x * z), \end{aligned}$$

and

$$\begin{aligned} x * (y * z) &= \alpha(x) \to (y * z) = \alpha(x) \to (\alpha(y) \to z) \\ &\overset{\text{(HH4)}}{=} \alpha(\alpha(x) \to y) \to (\alpha(x) \to z) \\ &= \alpha(x * y) \to (x * z) \\ &= (x * y) * (x * z). \end{aligned}$$

Thus $(H, *, 1)$ is a g-Hilbert algebra. □

We form a Hom-Hilbert algebra from a g-Hilbert algebra.

Theorem 3.5. *Let* $(H, *, 1)$ *be a* g*-Hilbert algebra and let* $\alpha : H \to H$ *be any bijective map. Define a binary operation* $\to$ *in* H *by* $x \to y := \alpha^{-1}(x) * y$ *for* $x, y \in H$*. Then the quadruple* $\mathcal{H} := (H, \to, 1, \alpha)$ *is a Hom-Hilbert algebra.*

Proof. Let $x, y, z \in H$. Since α is bijective, we get

$$\alpha(x) \to y = \alpha^{-1}(\alpha(x)) * y = x * y.$$

Then $\alpha(1) \to x = 1 * x \overset{\text{(gH1)}}{=} x$, $\alpha(x) \to x = x * x \overset{\text{(gH2)}}{=} 1$, and

$$\begin{aligned} \alpha(x) \to (\alpha(y) \to z) &= \alpha(x) \to (y * z) = x * (y * z) \overset{\text{(gH3)}}{=} y * (x * z) \\ &= \alpha(y) \to (x * z) = \alpha(y) \to (\alpha(x) \to z). \end{aligned}$$

Thus (HH1), (HH2), and (HH3) are valid. Finally, we have

$$\begin{aligned} \alpha(x) \to (\alpha(y) \to z) &= \alpha(x) \to (y * z) = x * (y * z) \\ &\overset{\text{(gH4)}}{=} (x * y) * (x * z) \\ &= (\alpha(x) \to y) * (\alpha(x) \to z) \\ &= \alpha(\alpha(x) \to y) \to (\alpha(x) \to z). \end{aligned}$$

So $\mathcal{H} := (H, \to, 1, \alpha)$ is a Hom-Hilbert algebra. □

The example below illustrates Theorem 3.5.

Example 3.6. Let $H = \{0, 1, a, b\}$ be a set with the Cayley table for $*$:

$*$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then $(H, *, 1)$ is a g-Hilbert algebra. Define a map $\alpha : H \to H$ by $\alpha(1) = 1$, $\alpha(a) = b$, $\alpha(b) = a$ and $\alpha(0) = 0$. Now, we apply our formula to generate the twisted Hom-implication presented by $x \to y := \alpha^{-1}(x) * y$ and its Cayley table is as follows:

$\to$	0	a	b	1
0	1	1	1	1
a	a	a	1	1
b	b	1	b	1
1	0	a	b	1

Thus $\mathcal{H} := (H, \to, 1, \alpha)$ is a Hom-Hilbert algebra.

Definition 3.7. A Hom-Hilbert algebra $\mathcal{H} := (H, \to, 1, \alpha)$ is said to be *commutative*, if it satisfies:

$$(\forall x, y \in H)(\alpha(\alpha(x) \to y) \to y = \alpha(\alpha(y) \to x) \to x). \tag{3.7}$$

Example 3.8. Let $H = \{1, a\}$ and define the binary operation $\to$ with the following Cayley table:

$\to$	1	a
1	1	1
a	1	a

Let $\alpha : H \to H$ be defined by $\alpha(1) = a$ and $\alpha(a) = 1$. Then $\mathcal{H} := (H, \to, 1, \alpha)$ is a commutative Hom-Hilbert algebra.

Proposition 3.9. *Every commutative Hom-Hilbert algebra $\mathcal{H} := (H, \to, 1, \alpha)$ satisfies for all $x, y \in H$:*

$$x \leq_\alpha y \implies y = \alpha(\alpha(y) \to x) \to x, \tag{3.8}$$

$$x \leq_\alpha \alpha(\alpha(x) \to y) \to y. \tag{3.9}$$

Proof. Suppose $x \leq_\alpha y$. Then $\alpha(x) \to y = 1$. Thus

$$\alpha(\alpha(y) \to x) \to x \overset{(3.7)}{=} \alpha(\alpha(x) \to y) \to y = \alpha(1) \to y \overset{(\mathrm{HH1})}{=} y.$$

Also, we have

$$\alpha(x) \to (\alpha(\alpha(x) \to y) \to y) \overset{(3.7)}{=} \alpha(\alpha(x) \to y) \to (\alpha(x) \to y) \overset{(\mathrm{HH2})}{=} 1.$$

So $x \leq_\alpha \alpha(\alpha(x) \to y) \to y$. □

Theorem 3.10. *If $\mathcal{H} := (H, \to, 1, \alpha)$ is a commutative Hom-Hilbert algebra, then $(H, \leq_\alpha)$ is a poset.*

Proof. Since the binary relation $\leq_\alpha$ is a preorder on H, it is sufficient to show that the binary relation $\leq_\alpha$ satisfies the antisymmetry. Suppose $x \leq_\alpha y$ and $y \leq_\alpha x$, i.e., $\alpha(x) \to y = 1$ and $\alpha(y) \to x = 1$. Then

$$y \overset{(3.8)}{=} \alpha(\alpha(y) \to x) \to x = \alpha(1) \to x \overset{\text{(HH1)}}{=} x.$$

Thus $(H, \leq_\alpha)$ is a poset. □

In a Hom-Hilbert algebra $\mathcal{H} := (H, \to, 1, \alpha)$, we define the twisted join, denoted as $\vee_\alpha$, as follows:

$$(\forall x, y \in H)(x \vee_\alpha y = \alpha(\alpha(x) \to y) \to y).$$

It is clear that if $\mathcal{H} := (H, \to, 1, \alpha)$ is a commutative Hom-Hilbert algebra, then $x \vee_\alpha y = y \vee_\alpha x$.

Theorem 3.11. *If* $\mathcal{H} := (H, \to, 1, \alpha)$ *is a commutative Hom-Hilbert algebra, then the map*

$$f : H \to H, \ x \mapsto x \vee_\alpha y$$

is $\leq_\alpha$*-monotone, and* $x \vee_\alpha y$ *is the least upper bound of* x *and* y *with respect to the partial order* $\leq_\alpha$*, and* $(H, \vee_\alpha, \leq_\alpha)$ *is a join-semilattice with top element* 1.

Proof. Suppose $y \leq_\alpha z$. Then $\alpha(z) \to x \leq_\alpha \alpha(y) \to x$ by (3.5), which implies from (3.5) again that

$$f(y) = y \vee_\alpha x = \alpha(\alpha(y) \to x) \to x \leq_\alpha \alpha(\alpha(z) \to x) \to x = z \vee_\alpha x = f(z).$$

Thus f is $\leq_\alpha$-monotone.

Let $x, y \in H$. Then $x \leq_\alpha x \vee_\alpha y$ by (3.9). Also, $y \leq_\alpha y \vee_\alpha x \overset{(3.7)}{=} x \vee_\alpha y$. Thus $x \vee_\alpha y$ is an upper bound of x and y with respect to the partial order $\leq_\alpha$. Let $z \in H$ be an upper bound of x and y with respect to the partial order $\leq_\alpha$, i.e., $x \leq_\alpha z$ and $y \leq_\alpha z$. Then $\alpha(\alpha(z) \to x) \to x = z$ by (3.8). Thus

$$x \vee_\alpha y = \alpha(\alpha(x) \to y) \to y = \alpha(\alpha(y) \to x) \to x \leq_\alpha \alpha(\alpha(z) \to x) \to x = z.$$

So $x \vee_\alpha y$ is the least upper bound of x and y with respect to the partial order $\leq_\alpha$. Since $(H, \leq_\alpha)$ is a poset (see Theorem 3.10), $(H, \vee_\alpha, \leq_\alpha)$ is a join-semilattice, and 1 is the top element since $x \vee_\alpha 1 = \alpha(\alpha(x) \to 1) \to 1 = 1$. □

Definition 3.12. A Hom-Hilbert algebra $\mathcal{H} := (H, \to, 1, \alpha)$ is said to be *bounded*, if there exists an element $0 \in H$ such that $0 \leq_\alpha x$, i.e., $\alpha(0) \to x = 1$, for all $x \in H$.

The bounded Hom-Hilbert algebra will be denoted as the quintuple $\mathcal{H}_0 := (H, \to, 1, 0, \alpha)$.

Example 3.13. Let $H = \{0, 1, a\}$ be a set with the Cayley table for →:

$\to$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

Define the twisting map $\alpha : H \to H$ by $\alpha(1) = 1$, $\alpha(a) = 0$, and $\alpha(0) = 0$. It is routine to check that $\mathcal{H}_0 := (H, \to, 1, 0, \alpha)$ is a bounded Hom-Hilbert algebra.

Let $\mathcal{H}_0 := (H, \rightarrow, 1, 0, \alpha)$ be a bounded Hom-Hilbert algebra. Define a unary operation

$$\neg_\alpha : H \rightarrow H,\ x \mapsto \alpha(x) \rightarrow 0.$$

Proposition 3.14. *Every bounded Hom-Hilbert algebra* $\mathcal{H}_0 := (H, \rightarrow, 1, 0, \alpha)$ *satisfies the following assertions for all* $x, y \in H$*:*

$$x \vee_\alpha 0 = x, \tag{3.10}$$

$$\neg_\alpha(0) = 1,\ \neg_\alpha(1) = 0, \tag{3.11}$$

$$\alpha(0) \rightarrow \neg_\alpha(x) = 1, \tag{3.12}$$

$$\neg_\alpha(\neg_\alpha(x)) = x, \tag{3.13}$$

$$x \leq_\alpha y \implies \neg_\alpha(y) \leq_\alpha \neg_\alpha(x). \tag{3.14}$$

Proof. (3.10): If we take $y = 0$ in (3.7), then

$$x \vee_\alpha 0 = \alpha(\alpha(x) \rightarrow 0) \rightarrow 0 = \alpha(\alpha(0) \rightarrow x) \rightarrow x = \alpha(1) \rightarrow x \overset{\text{(HH1)}}{=} x$$

for all $x \in H$.

(3.11): $\neg_\alpha(0) = \alpha(0) \rightarrow 0 \overset{\text{(HH2)}}{=} 0$ and $\neg_\alpha(1) = \alpha(1) \rightarrow 0 \overset{\text{(HH1)}}{=} 0$.

(3.12): For every $x \in H$, we get

$$\begin{aligned}\alpha(0) \rightarrow \neg_\alpha(x) = \alpha(0) \rightarrow (\alpha(x) \rightarrow 0) &\overset{\text{(HH3)}}{=} \alpha(x) \rightarrow (\alpha(0) \rightarrow 0)\\ &\overset{\text{(HH2)}}{=} \alpha(x) \rightarrow 1 \overset{(3.1)}{=} 1.\end{aligned}$$

(3.13): If we take $y = 0$ in (3.7), then

$$\begin{aligned}\neg_\alpha(\neg_\alpha(x)) &= \alpha(\alpha(x) \rightarrow 0) \rightarrow 0 = \alpha(\alpha(0) \rightarrow x) \rightarrow x\\ &= \alpha(1) \rightarrow x \overset{\text{(HH1)}}{=} x.\end{aligned}$$

(3.14): Suppose $x \leq_\alpha y$. Then $\alpha(x) \rightarrow y = 1$. Thus

$$\begin{aligned}\alpha(\neg_\alpha(y)) \rightarrow \neg_\alpha(x) &= \alpha(\alpha(y) \rightarrow 0) \rightarrow (\alpha(x) \rightarrow 0)\\ &\overset{\text{(HH3)}}{=} \alpha(x) \rightarrow (\alpha(\alpha(y) \rightarrow 0) \rightarrow 0)\\ &= \alpha(x) \rightarrow (\neg_\alpha(\neg_\alpha(y)))\\ &\overset{(3.13)}{=} \alpha(x) \rightarrow y = 1.\end{aligned}$$

So $\neg_\alpha(y) \leq_\alpha \neg_\alpha(x)$. □

Let $\mathcal{H}_0 := (H, \rightarrow, 1, 0, \alpha)$ be a bounded Hom-Hilbert algebra. Using the twisted join $\vee_\alpha$, we define the twisted meet, denoted by $\wedge_\alpha$, as follows:

$$(\forall x, y \in H)(x \wedge_\alpha y = \neg_\alpha(\neg_\alpha(x) \vee_\alpha \neg_\alpha(y))).$$

Using the Commutativity of the twisted join $\vee_\alpha$, it is clear that if $\mathcal{H}_0 := (H, \rightarrow, 1, 0, \alpha)$ is a bounded commutative Hom-Hilbert algebra, then $x \wedge_\alpha y = y \wedge_\alpha x$ for all $x, y \in H$.

Theorem 3.15. *Let* $\mathcal{H}_0 := (H, \rightarrow, 1, 0, \alpha)$ *be a bounded Hom-Hilbert algebra. Then the twisted meet* $x \wedge_\alpha y$ *is the greatest lower bound of* x *and* y *with respect to the partial order* $\leq_\alpha$*, and* $(H, \wedge_\alpha, \leq_\alpha)$ *is a meet-semilattice with the bottom element* 0.

Proof. Using Theorem 3.11, we know that $\neg_\alpha(x) \vee_\alpha \neg_\alpha(y)$ is the least upper bound of $\neg_\alpha(x)$ and $\neg_\alpha(y)$, i.e., $\neg_\alpha(x) \leq_\alpha \neg_\alpha(x) \vee_\alpha \neg_\alpha(y)$ and $\neg_\alpha(y) \leq_\alpha \neg_\alpha(x) \vee_\alpha \neg_\alpha(y)$. It follows from (3.14) and (3.13) that

$$x \wedge_\alpha y = \neg_\alpha(\neg_\alpha(x) \vee_\alpha \neg_\alpha(y)) \leq_\alpha \neg_\alpha(\neg_\alpha(x)) = x$$

and $x \wedge_\alpha y = \neg_\alpha(\neg_\alpha(x) \vee_\alpha \neg_\alpha(y)) \leq_\alpha \neg_\alpha(\neg_\alpha(y)) = y$. Then $x \wedge_\alpha y$ is a lower bound of x and y with respect to the partial order $\leq_\alpha$. Let z be any other lower bound of x and y with respect to the partial order $\leq_\alpha$, i.e., $z \leq_\alpha x$ and $z \leq_\alpha y$. Then $\neg_\alpha(x) \leq_\alpha \neg_\alpha(z)$ and $\neg_\alpha(y) \leq_\alpha \neg_\alpha(z)$ by (3.14). Thus $\neg_\alpha(z)$ is an upper bound of $\neg_\alpha(x)$ and $\neg_\alpha(y)$. So $\neg_\alpha(x) \vee_\alpha \neg_\alpha(y) \leq_\alpha \neg_\alpha(z)$ since $\neg_\alpha(x) \vee_\alpha \neg_\alpha(y)$ is the least upper bound of $\neg_\alpha(x)$ and $\neg_\alpha(y)$. Usaing (3.13) and (3.14) yields $z = \neg_\alpha(\neg_\alpha(z)) \leq_\alpha \neg_\alpha(\neg_\alpha(x) \vee_\alpha \neg_\alpha(y)) = x \wedge_\alpha y$. Hence $x \wedge_\alpha y$ is the greatest lower bound of x and y with respect to the partial order $\leq_\alpha$. Since $(H, \leq_\alpha)$ is a poset (See Theorem 3.10), $(H, \wedge_\alpha, \leq_\alpha)$ is a meet-semilattice with the bottom element 0. □

The combination of Theorem 3.10, Theorem 3.11 and Theorem 3.15 yields the following theorem.

Theorem 3.16. *If $\mathcal{H}_0 := (H, \to, 1, 0, \alpha)$ is a bounded commutative Hom-Hilbert algebra, then the quadruple $(H, \leq_\alpha, \vee_\alpha, \wedge_\alpha)$ is a bounded lattice.*

Proposition 3.17. *Every bounded commutative Hom-Hilbert algebra $\mathcal{H}_0 := (H, \to, 1, 0, \alpha)$ satisfies the following items: for all $x, y, z \in H$,*

$$\neg_\alpha(x) \vee_\alpha y = \alpha(x) \to y, \tag{3.15}$$

$$z \wedge_\alpha x \leq_\alpha y \iff z \leq_\alpha \alpha(x) \to y, \tag{3.16}$$

$$x \wedge_\alpha (y \vee_\alpha z) = (x \wedge_\alpha y) \vee_\alpha (x \wedge_\alpha z), \tag{3.17}$$

$$x \vee_\alpha (y \wedge_\alpha z) = (x \vee_\alpha y) \wedge_\alpha (x \vee_\alpha z). \tag{3.18}$$

Proof. (3.15): For every $x, y \in H$, we have

$$\begin{aligned}
\neg_\alpha(x) \vee_\alpha y &= \alpha(\alpha(\neg_\alpha(x)) \to y) \to y \\
&\overset{(3.7)}{=} \alpha(\alpha(y) \to \neg_\alpha(x)) \to \neg_\alpha(x) \\
&= \alpha(\alpha(y) \to (\alpha(x) \to 0)) \to (\alpha(x) \to 0) \\
&\overset{(\text{HH3})}{=} \alpha(\alpha(x) \to (\alpha(y) \to 0)) \to (\alpha(x) \to 0) \\
&= \alpha(\alpha(x) \to \neg_\alpha(y)) \to (\alpha(x) \to 0) \\
&\overset{(\text{HH4})}{=} \alpha(x) \to (\alpha(\neg_\alpha(y)) \to 0) \\
&= \alpha(x) \to \neg_\alpha(\neg_\alpha(y)) \\
&\overset{(3.13)}{=} \alpha(x) \to y.
\end{aligned}$$

(3.16): If $z \wedge_\alpha x \leq_\alpha y$, then $\neg_\alpha(\neg_\alpha(z) \vee_\alpha \neg_\alpha(x)) \leq_\alpha y$. Thus

$$\neg_\alpha(y) \overset{(3.14)}{\leq_\alpha} \neg_\alpha(\neg_\alpha(\neg_\alpha(z) \vee_\alpha \neg_\alpha(x))) \overset{(3.13)}{=} \neg_\alpha(z) \vee_\alpha \neg_\alpha(x).$$

So

$$\begin{aligned}1 &= \alpha(\neg_\alpha(y)) \to (\neg_\alpha(z) \vee_\alpha \neg_\alpha(x))\\ &\overset{(3.15)}{=} \alpha(\neg_\alpha(y)) \to (\alpha(z) \to \neg_\alpha(x))\\ &\overset{(HH3)}{=} \alpha(z) \to (\alpha(\neg_\alpha(y)) \to \neg_\alpha(x))\\ &\overset{(3.15)}{=} \alpha(z) \to (\neg_\alpha(\neg_\alpha(y)) \vee_\alpha \neg_\alpha(x))\\ &\overset{(3.13)}{=} \alpha(z) \to (y \vee_\alpha \neg_\alpha(x))\\ &= \alpha(z) \to (\neg_\alpha(x) \vee_\alpha y)\\ &\overset{(3.15)}{=} \alpha(z) \to (\alpha(x) \to y).\end{aligned}$$

Hence $z \leq_\alpha \alpha(x) \to y$.

Conversely, suppose $z \leq_\alpha \alpha(x) \to y$. Let $\mathfrak{b} = z \wedge_\alpha x$. Then $\mathfrak{b} \leq_\alpha z$ and $\mathfrak{b} \leq_\alpha x$. Thus $\mathfrak{b} \leq_\alpha \alpha(x) \to y$ since $\leq_\alpha$ is transitive. So

$$\begin{aligned}1 = \alpha(\mathfrak{b}) \to (\alpha(x) \to y) &\overset{(HH4)}{=} \alpha(\alpha(\mathfrak{b}) \to x) \to (\alpha(\mathfrak{b}) \to y)\\ = \alpha(1) \to (\alpha(\mathfrak{b}) \to y) &\overset{(HH1)}{=} \alpha(\mathfrak{b}) \to y,\end{aligned}$$

Hence $z \wedge_\alpha x = \mathfrak{b} \leq_\alpha y$.

(3.17): Let $x, y, z \in H$. Since $x \wedge_\alpha y \leq_\alpha x$ and $x \wedge_\alpha y \leq_\alpha y \leq_\alpha y \vee_\alpha z$, the transitivity of $\leq_\alpha$ gives $x \wedge_\alpha y \leq_\alpha y \vee_\alpha z$. Then $x \wedge_\alpha y \leq_\alpha x \wedge_\alpha (y \vee_\alpha z)$. By the similar way, $x \wedge_\alpha z \leq_\alpha x \wedge_\alpha (y \vee_\alpha z)$. It follows that

$$(x \wedge_\alpha y) \vee_\alpha (x \wedge_\alpha z) \leq_\alpha x \wedge_\alpha (y \vee_\alpha z).$$

Let $R = (x \wedge_\alpha y) \vee_\alpha (x \wedge_\alpha z)$. Then $x \wedge_\alpha y \leq_\alpha R$ and $x \wedge_\alpha z \leq_\alpha R$. Thus $y \leq_\alpha \alpha(x) \to R$ and $z \leq_\alpha \alpha(x) \to R$ by (3.16), i.e., $\alpha(x) \to R$ is an upper bound of y and z. So $y \vee_\alpha z \leq_\alpha \alpha(x) \to R$ which is equivalent to

$$x \wedge_\alpha (y \vee_\alpha z) \leq_\alpha R = (x \wedge_\alpha y) \vee_\alpha (x \wedge_\alpha z).$$

Hence $x \wedge_\alpha (y \vee_\alpha z) = (x \wedge_\alpha y) \vee_\alpha (x \wedge_\alpha z)$.

(3.18): It is obtained in a similar way to (3.17). □

Theorem 3.18. *If $\mathcal{H}_0 := (H, \to, 1, 0, \alpha)$ is a bounded commutative Hom-Hilbert algebra, then the quadruple $(H, \leq_\alpha, \vee_\alpha, \wedge_\alpha)$ is a distributive bounded lattice.*

Proof. This is an immediate result of Theorem 3.16 and Proposition 3.17. □

Lemma 3.19. *Let $\mathcal{H}_0 := (H, \to, 1, 0, \alpha)$ be a bounded commutative Hom-Hilbert algebra. Then $(H, \wedge_\alpha, 1)$ and $(H, \vee_\alpha, 0)$ are commutative monoids. If we define $x \Rightarrow y := \alpha(x) \to y$ for all $x, y \in H$, then $z \wedge_\alpha x \leq_\alpha y \iff z \leq_\alpha x \Rightarrow y$.*

Proof. For every $x, y, z \in H$, let $\mathfrak{c} = (x \wedge_\alpha y) \wedge_\alpha z$. Then $\mathfrak{c}$ is a lower bound of $x \wedge_\alpha y$ and z. Since $\mathfrak{c} \leq_\alpha x \wedge_\alpha y$, it is also a lower bound of x and y. Thus $\mathfrak{c} \leq_\alpha x$, $\mathfrak{c} \leq_\alpha y$ and $\mathfrak{c} \leq_\alpha z$. Also, $\mathfrak{c} \leq_\alpha (y \wedge_\alpha z)$, since $\mathfrak{c}$ is a lower bound of y and z, so $(x \wedge_\alpha y) \wedge_\alpha z = \mathfrak{c} \leq_\alpha x \wedge_\alpha (y \wedge_\alpha z)$. By applying the exact same logic in reverse, we get $x \wedge_\alpha (y \wedge_\alpha z) \leq_\alpha (x \wedge_\alpha y) \wedge_\alpha z$. Using the antisymmetry of $\leq_\alpha$ yields $x \wedge_\alpha (y \wedge_\alpha z) = (x \wedge_\alpha y) \wedge_\alpha z$. It is clear that $x \wedge_\alpha 1 \leq_\alpha x$. Since

$x \leq_\alpha 1$ and $x \leq_\alpha x$, i.e., x is a lower bound x and 1, we have $x \leq_\alpha x \wedge_\alpha 1$. Thus $x \wedge_\alpha 1 = x$. Since the twisted meet $\wedge_\alpha$ is commutative, it also follows that $1 \wedge_\alpha x = x$. So $(H, \wedge_\alpha, 1)$ is a commutative monoid. By the similar way, we can verify that $(H, \vee_\alpha, 0)$ is a commutative monoid. If $x \Rightarrow y \overset{\text{def}}{=} \alpha(x) \to y$ for all $x, y \in H$, then $z \wedge_\alpha x \leq_\alpha y \iff z \leq_\alpha x \Rightarrow y$ for all $x, y, z \in H$ by (3.16). □

By comprehensively considering the above discussion, we derive the following theorem.

Theorem 3.20. *If $\mathcal{H}_0 := (H, \to, 1, 0, \alpha)$ is a bounded commutative Hom-Hilbert algebra, then $(H, \leq_\alpha, \wedge_\alpha, \vee_\alpha, \Rightarrow, 0, 1)$ is a commutative residuated lattice.*

References

[1] A. A. A. Agboola, M. A. Ibrahim, A. O. Adeniji and S. A. Adebisi, On fundamental properties of Hom-groups, J. Mahani Math. Res. 14 (2) (2025) 81–97. https://doi.org/10.22103/jmmr.2024.24063.1697

[2] Z. Chebel, H. Adimi and H. Bouremel, Hom-actions and class equation for Hom-groups, Journal of Geometry and Physics 207 (2025) 105371. https://doi.org/10.1016/j.geomphys.2024.105371

[3] M. Hassanzadeh, Lagrange's theorem for Hom-groups, Rocky Mountain Journal of Mathematics 49 (3) (2019) 773–787.

[4] A. Makhlouf and S. D. Silvestrov, Hom-algebra structures, Journal of Generalized Lie Theory and Applications 2 (2) (2008) 51–64. https://www.hilarispublisher.com/open-access/homalgebra-structures-1736-4337-2-115.pdf

[5] Y. Sheng and D. Chen, Hom-Lie2-algebras, Journal of Algebra 376 (2013) 174–195. http://dx.doi.org/10.1016/j.jalgebra.2012.11.032

[6] A. Diego, Sur les algébres de Hilbert, Collection de Logique Mathématique, Serie A, 21, Gauthier-Villars, Paris 1966.

[7] R. A. Borzooei and J. Shohani, On generalized Hilbert algebras, Italian Journal of Pure and Applied Mathematic 29 (2012) 71–86.

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